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ON THE IDENTIFICATION OF SYSTEMS AND THE UNSCRAMBLING OF DATA–II: AN INVERSE PROBLEM IN RADIATIVE TRANSFER

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PREFACE AND SUMMARY

The advent of artificial satellites makes it desirable to be able to use measurements of the angular dependence of diffusely reflected sunlight from a planetary atmosphere to deduce the physical properties of the atmosphere. In this Memorandum it is shown that a combination of invariant imbedding and quasilinearization may be used for this purpose. Some illustrative numerical results are provided.

1. Introduction. In the first paper in this series,[1] we presented a general formulation of a significant class of inverse problems and outlined the use of quasilinearization as a systematic procedure for obtaining numerical solutions. In this paper we wish to indicate the application of these techniques to some important questions in radiative transfer. Our investigations are aimed at the complex problem of determining the nature of planetary atmospheres on the basis of various observations of the angular dependence of diffusely reflected light, and are pertinent to the design of experiments in this area. The methods, however, are applicable to a wide class of physical processes.

2. Statement of Problem. Consider an inhomogeneous, plane-parallel, non-emitting and isotropically scattering atmosphere of finite optical thickness τ_1 whose optical properties depend only upon τ, the optical height above the bottom; see Fig. 1.

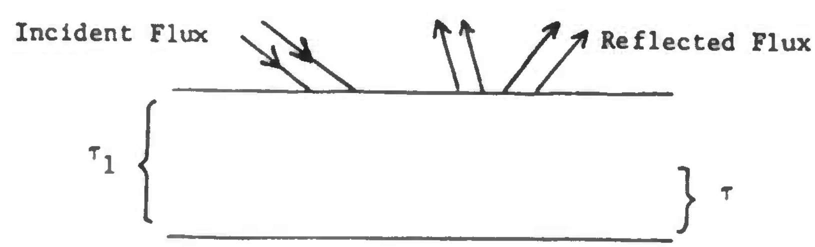

FIG. 1. THE PHYSICAL SITUATION

Let parallel rays of light of net flux π per unit area normal to their direction of propagation be incident on the upper surface in the direction characterized by μ_o, where μ_o is, as usual, the cosine of the angle measured from the inward normal to the surface. We suppose here that the bottom surface is a completely absorbing barrier. This is not an essential requirement.

Let us now employ invariant imbedding.[2-4] Let $r(\mu,\mu_o,\tau_1)$ denote the intensity of the diffusely reflected light in the direction $\cos^{-1}\mu$ and set $R(\mu,\mu_o,\tau_1) = 4\mu r$. Then R satisfies the integro-differential equation

$$\frac{\partial R}{\partial \tau_1} = -\left(\mu^{-1} + \mu_o^{-1}\right) R + \lambda(\tau_1) \left[1 + \frac{1}{2} \int_0^1 R(\mu,\mu',\tau_1) \frac{d\mu'}{\mu'}\right] \times$$
$$\times \left[1 + \frac{1}{2} \int_0^1 R(\mu',\mu_o,\tau_1) \frac{d\mu'}{\mu'}\right] \tag{1}$$

with the initial condition $R(\mu,\mu_o,0) = 0$. The function $\lambda(\tau)$ is the albedo for single scattering.

The inverse problem we wish to consider is that of determining the nature of $\lambda(\tau)$ from measurements of the reflected flux at various angles.

3. <u>Analytic Formulation</u>. In order to do this, we must impose some analytical structure upon $\lambda(\tau)$, which is to say, we must add some information concerning the nature of the physical process. To illustrate how this is done, consider the case where the medium consists essentially of two layers, each with constant albedo, separated by a thin zone of rapid transition from one value of the albedo to the other. Let the albedo have the form

$$\lambda(\tau) = a + b \tanh 10(\tau - c) \tag{2}$$

so that $\lambda_1 \cong a - b$ in Layer 1 and $\lambda_2 \cong a + b$ in Layer 2.

In place of Eq. (2), let us use the discrete equations obtained from Gauss quadrature,[2-4]

$$\frac{\partial R_{ij}(\tau_1)}{\partial \tau_1} = -\left(\mu_i^{-1} + \mu_j^{-1}\right) R_{ij} + \lambda(\tau_1)\left[1 + \frac{1}{2}\sum_{k=1}^{N} R_{ik}(\tau_1)\frac{w_k}{\mu_k}\right] \times \left[1 + \frac{1}{2}\sum_{k=1}^{N} R_{kj}(\tau_1)\frac{w_k}{\mu_k}\right] \tag{3}$$

Using these equations, we generate "observations" by choosing $a = .5$, $b = .1$, $c = .5$, and integrating to a thickness of $\tau_1 = 1.0$. We also use $N = 7$.

Starting with the values $b_{ij} \approx r_{ij}(1)$, we want to determine the quantities a, b, c, and τ_1, the thickness. To do this, we ask for the values a, b, c and τ_1 which minimize the expression

$$S = \sum_{i,j} \left\{r_{ij}(\tau_1) - b_{ij}\right\}^2, \tag{4}$$

where $R_{ij}(\tau_1) = 4\mu_i r_{ij}(\tau_1)$ is the solution of Eq. (3).

The method we use is that of quasilinearization, as outlined in Ref. 1 and presented in detail in Refs. 5-7.

4. Numerical Results. We carried out three types of numerical experiments:

a. Determine c, the altitude of the interface, given a, b, and τ_1.

b. Determine τ_1, the overall thickness, given a, b, and c.

c. Determine a, b, and c, the two albedos and the altitude of the interface, given τ_1.

Following is a brief tabular indication of the results. As will be seen, the success of the method depends crucially upon the initial approximation. In other words, we can only expect the method to be

successful if we have at least some rough idea of the nature of the actual physical process.

Table 1

SUCCESSIVE APPROXIMATIONS OF c, THE LEVEL OF THE INTERFACE

Approximation	Run 1	Run 2	Run 3
0	0.2	0.8	0.0
1	0.62	0.57	
2	0.5187	0.5024	No
3	0.500089	0.499970	convergence
4	0.499990	0.499991	
True Value	0.5	0.5	0.5

Table 2

SUCCESSIVE APPROXIMATIONS OF λ_1, λ_2, AND c

Approximation	λ_1=a-b	λ_2=a+b	c
0	0.51	0.69	0.4
1	0.4200	0.6052	0.5038
2	0.399929	0.599995	0.499602
3	0.399938	0.599994	0.499878
True Value	0.4	0.6	0.5

The latter table is based on knowing the reflected intensity for seven angles of reflection for each of seven angles of incidence. The quasilinearization technique involves integrating 124 linear differential equations at each stage and solving a system of linear algebraic equations of order three. The calculations took about two minutes on an IBM 7044.

Future studies involve determining the accuracies in the intensity measurements required to yield specified accuracies in the estimates of the properties of the medium. Experiments for a quadratic profile for the albedo function will be reported upon in the near future.

REFERENCES

1. Bellman, R. E., H. H. Kagiwada, and R. E. Kalaba, On the Identification of Systems and the Unscrambling of Data-I: Hidden Periodicities, The RAND Corporation, RM-4285-PR, September 1964.

2. Chandrasekhar, S., Radiative Transfer, Dover Publications, Inc., New York, 1950.

3. Bellman, R. E., R. E. Kalaba, and M. C. Prestrud, Invariant Imbedding and Radiative Transfer in Slabs of Finite Thickness, American Elsevier Publishing Company, New York, 1963.

4. Bellman, R. E., H. H. Kagiwada, R. E. Kalaba, and M. C. Prestrud, Invariant Imbedding and Time-Dependent Transport Processes, American Elsevier Publishing Company, New York, 1964.

5. Kalaba, R. E., "On Nonlinear Differential Equations, the Maximum Operation and Monotone Convergence," J. Math. Mech., Vol. 8, No. 4, July 1959, pp. 519-574.

6. Bellman, R. E., and R. E. Kalaba, Quasilinearization and Boundary Value Problems, American Elsevier Publishing Company, New York, to appear.

7. Bellman, R. E., H. H. Kagiwada, and R. E. Kalaba, "Orbit Determination as a Multi-point Boundary Value Problem and Quasilinearization," Proc. Nat. Acad. Sci. USA, Vol. 48, No. 8, August 1962, pp. 1327-1329.

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